The generalized discrete $(r|p)$-centroid problem

D.R. Santos-Peñate[a,*], C.M. Campos-Rodríguez[b], J.A. Moreno-Pérez[b]

[a] *Instituto Universitario de Turismo y Desarrollo Económico Sostenible, Universidad de Las Palmas de G.C., 35017. Las Palmas de Gran Canaria. Spain*
[b] *Instituto Universitario de Desarrollo Regional. University of La Laguna. 38271. La Laguna. Spain*

Abstract

The $(r|p)$-centroid problem or leader-follower problem is generalized considering different customer choice rules where a customer may use facilities belonging to different firms if the difference in travel distance (or time) is small enough. Assuming essential goods, some particular customer choice rules are analyzed. Linear programming formulations for the generalized $(r|X_p)$-medianoid and $(r|p)$-centroid problems are presented and an exact solution approach is applied. Some computational examples are included.

Keywords: competitive location, bi-level problems, $(r|p)$-centroid, $(r|X_p)$-medianoid, leader-follower problem, linear programming

1. Introduction

The $(r|p)$-centroid problem is a competitive location problem where two players, the leader and the follower, enter the market sequentially and compete in providing goods and services to customers. The leader enters the market first with p facilities and seeks to minimize the maximum market share captured by a future competitor, called the follower. The follower opens r facilities at the locations that maximize its market share. We consider the case of essential goods, which means that demand has to be satisfied, and so customers will visit at least one facility to obtain all the goods and services they need. As demand is assumed to be essential, the objective of minimizing the maximum market share the competitor can capture is equivalent to maximizing one's own market share.

☆This research is partially financed by the Spanish Government, grant ECO2014-59067-P and TIN2012-32608

*Corresponding author

Email addresses: `dr.santos@ulpgc.es` (D.R. Santos-Peñate), `ccampos@ull.es` (C.M. Campos-Rodríguez), `jamoreno@ull.es` (J.A. Moreno-Pérez)

The customer choice rule represents the behaviour of the customers. The binary rule represents the "all or nothing" behaviour, according to which a customer uses the closest facility, disregarding any other facility that is more distant, even if the difference in terms of distance is very small. The ties among the competing firms are solved by a sharing function. The binary choice rule assumes that customers are sensitive to any difference in distances to the facilities. Under the partially binary choice rule, a customer visits the closest facility for both firms. In this case, a customer could visit a facility belonging to firm A and pass over a closer facility belonging to firm B but not the closest one to the customer. According to the proportional choice rule, a customer visits all the facilities and the proportion of the demand captured depends on the travel distance. Binary, partially binary and proportional rules, for essential and unessential demands, are studied in Hakimi (1990). Some customer choice rules replace the hyper-sensitive consumer conduct implicit in the binary model by a threshold-sensitive behaviour; in this case, a customer only uses firm A exclusively if the distance from this customer to the competitors exceeds the distance to firm A by an amount greater than or equal to a threshold or *minimum sensibility* (Devletoglou, 1965; Devletoglou and Demetriou, 1967). An alternative to the binary and proportional rules is a threshold-sensitive choice rule, under which the demand captured by each firm in the *doubtful zone* is given by a non-increasing function of the travel distance, such as the decay functions used in the generalized coverage models (Berman et al., 2003, 2010; Berman and Krass, 2002).

The $(r|p)$-centroid problem is a bi-level problem whose resolution, even for a moderate size, requires significant computational effort. Some solution approaches can be found in the literature. An exact algorithm to find the locations that maximize the expected profit is presented in Gosh and Craig (1984). However, this approach consists basically of an enumeration of the feasible solutions for the leader, and so this algorithm is not very useful. A tabu search algorithm is proposed in Benati and Laporte (1994), and Davydov et al. (2014). In Campos Rodríguez et al. (2010), the $(r|p)$-centroid is solved via an exact algorithm based on the evaluation of the score (demand captured by the best locations for the follower) of a sequence of leader's solutions constrained to a family of good follower's solutions. During the process, the leader's solutions with a score higher than the current upper bound of the optimum are eliminated from the feasible set. Another exact algorithm is presented in Aleekseeva et al. (2010). In this case, an alternating heuristic, used previously in Bhadury et al. (2003) to solve the centroid problem in the plane, is applied to obtain initial solutions. At each iteration, the problem of the leader constrained to a family of follower's solutions is solved to obtain a lower bound of the optimum; then, for the leader's solution obtained, the problem of the follower is solved to obtain an upper bound. The process ends when the best lower and upper bounds coincide. A branch-and-cut algorithm to solve the $(r|p)$-centroid problem is proposed in Rodoredo and Pessoa

(2013). A variable neighbourhood search is used in Davydov et al. (2014). Other heuristic and exact methods to solve the discrete $(r|p)$-centroid problem are described in Alekseeva and Kochetov (2013). Most of the mentioned references consider the binary and essential scenario. Different scenarios are analyzed in Biesinger et al. (2015a, 2015b), where the problem is solved by applying evolutionary algorithms.

In this paper we consider a general customer choice rule which incorporates the possibility of a customer visiting both firms if the distances to the competing firms are not very different. This generalized choice function includes the binary rule as a particular case. We use this general choice function to define the generalized discrete $(r|p)$-centroid problem in the case of essential goods, and analyze some particular decay functions. Linear programming formulations for both problems (follower and leader), are given, and an exact procedure is applied.

The remainder of this paper is organized as follows. The model is introduced in Section 2 and an example is then described in Section 3. Integer programming formulations are presented in Section 4. A solution approach is described in Section 5. Some computational examples are presented in Section 6 and, finally, the main conclusions drawn are summarized in Section 7.

2. Problem statement

Let $C = \{c_k : k \in [1..n]\}$ be a finite set of demand points or clients and $L = \{l_i : i \in [1..m]\}$ be a finite set of potential locations for facilities. Every point $c_k \in C$ has a weight $w_k = w(c_k)$ which represents the demand at c_k. Let the total demand be $W_T = \sum_{k=1}^{n} w_k$. Let $d_{ki} = d(c_k, l_i)$ be the distance between point $c_k \in C$ and point $l_i \in L$, and, for $c_k \in C$ and $X \subseteq L$, $d_{kX} = \min_{x \in X} d(c_k, x)$ is the distance between c_k and X.

We consider a market of essential goods, which means that the demand is totally satisfied, that is, the sum of demands served by the firms operating in the market is equal to the total existing demand.

Assume that two competing firms, A and B, operate in the market with p and r facilities located at $X_p \subset L$ and $Y_r \subset L$, respectively. The demand at point c_k captured by the firms depends on the difference $\delta_k = d_{kY_r} - d_{kX_p}$. The market shares for firms A and B are given, respectively, by $W_A = W_T - W_B$ and

$$W_B = W_B(X_p, Y_r) = \sum_{k=1}^{n} w_k f_k(\delta_k) \tag{1}$$

where $f_k(\delta)$ is a non-negative and non-increasing function such that $0 \leq f_k(\delta) \leq 1$ for $\delta \geq 0$.

Table 1 shows some different capture functions. For simplicity in the notation, index k has been eliminated from the table. The piecewise linear, piecewise concave and piecewise convex functions, for three pieces, are plotted in Figure 1.

Table 1: Particular capture functions

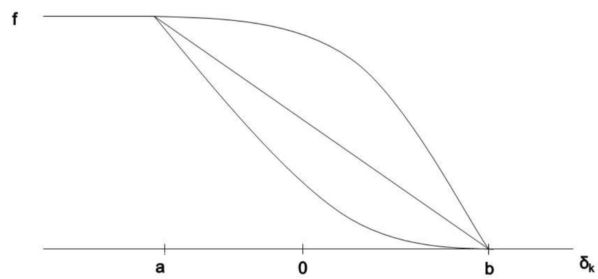
Binary	$f(\delta) = \begin{cases} 1 & \text{if} \quad \delta < 0 \\ \mu & \text{if} \quad \delta = 0 \\ 0 & \text{if} \quad \delta > 0 \end{cases}$, where $\mu \in [0, 1]$.
Step	$f(\delta) = \begin{cases} 1 & \text{if} \quad \delta \leq \delta_1 \\ a_q & \text{if} \quad \delta_q < \delta \leq \delta_{q+1},\ 1 \leq q \leq Q \\ 0 & \text{if} \quad \delta > \delta_{Q+1} \end{cases}$, where $1 > a_1 > \cdots > a_Q > 0$.
Continuous piecewise linear	$f(\delta) = \begin{cases} 1 & \text{if} \quad \delta \leq \delta_1 \\ a_q\delta + b_q & \text{if} \quad \delta_q < \delta \leq \delta_{q+1},\ 1 \leq q \leq Q \\ 0 & \text{if} \quad \delta > \delta_{Q+1} \end{cases}$, where $\begin{cases} 1 & = & a_1\delta_1 + b_1 \\ a_q\delta_q + b_q & = & a_{q-1}\delta_q + b_{q-1},\ 1 < q < Q \\ 0 & = & a_Q\delta_{Q+1} + b_Q \end{cases}$.
Continuous piecewise concave	For example, quadratic: $f(\delta) = \begin{cases} 1 & \text{if} \quad \delta \leq a \\ 1 - \left(\frac{\delta - a}{b - a}\right)^2 & \text{if} \quad a < \delta \leq b \\ 0 & \text{if} \quad \delta > b \end{cases}$, where $a < b$.
Continuous piecewise convex	For example, quadratic: $f(\delta) = \begin{cases} 1 & \text{if} \quad \delta \leq a \\ \left(\frac{b - \delta}{b - a}\right)^2 & \text{if} \quad a < \delta \leq b \\ 0 & \text{if} \quad \delta > b \end{cases}$ where $a < b$.

Initially, no firm is operating in the market, firm A, the leader, wants to enter the market with p facilities taking into account that firm B, the follower, will enter the market later, installing r facilities at the locations where the market share of B is maximum. Firm A wants to determine the p locations that minimize the maximum demand that firm B can capture.

If the leader has p facilities open at X_p, the problem of the follower is to determine the set Y_r of r locations that maximize its market share $W_B(X_p, Y_r)$. An optimal solution to this problem, $Y_r(X_p)$, is an $(r|X_p)$-medianoid. The problem of the leader is to determine the set X_p that minimizes $W_B(X_p, Y_r(X_p))$, that is, the set X_p which minimizes the maximum market share that the follower could achieve. An optimal solution to the problem of the leader is an $(r|p)$-centroid. Let $L^p = \{X \subseteq L : |X| = p\}$, for any natural number $p > 0$. Then, formally, the $(r|p)$-centroid problem or leader's problem is the following minimax problem

$$\min_{X \in L^p} \max_{Y \in L^r} W_B(X, Y). \tag{2}$$

Figure 1: Examples of decay functions



That is,

$$\min_{X \in L^p} S(X) \text{ where } S(X) = \max_{Y \in L^r} W_B(X, Y). \tag{3}$$

For $X \subset L$, with $|X| = p$, $S(X)$ is the score of X. Problem (2) (or (3)) is a bi-level problem where the lower level problem is the $(r|X_p)$-medianoid problem and the upper level problem is the $(r|p)$-centroid problem.

2.1. Some results

In this section, results are obtained for some common capture functions shown in Table 1 and the case $p = r$. The propositions show situations in which the set of optimal locations for the follower is the same as the set of optimal locations for the leader. In particular, Proposition 4 shows that, if we consider the piecewise linear functions, under certain conditions, the p-median is the solution for both the leader and the follower. In this case, the p-median, X_M, is a $(p|p)$-centroid and a $(p|X_M)$-medianoid.

Proposition 1. *Let, for every k,*

$$f_k(\delta) = f(\delta; a, b) = \begin{cases} 1 & \text{if} \quad \delta \le a \\ \frac{b-\delta}{b-a} & \text{if} \quad a < \delta \le b \\ 0 & \text{if} \quad \delta > b \end{cases}$$

where $a \le 0 < b$. If, for $a = a_0,\ b = b_0$, set X is a $(p|p)$-centroid with $Y(X) = X$, then for all $0 \ge a \ge a_0$ and $b \ge b_0 > 0$, we have $Y(X) = X$ and X is a $(p|p)$-centroid.

Proof. Consider $0 \geq a_1 \geq a_0$, $b_1 \geq b_0 > 0$, and the customer choice rule defined by $f_k(\delta) = f(\delta; a_t, b_t)$, for $t = 0, 1$ and for all k.

For $Y \in L^p$ denote $\delta_k = d_{kY} - d_{kX}$ and define $\Delta_t = W_B(X, Y) - W_B(X, X)$ for $t = 0, 1$. That is,

$$\Delta_t = \Big(1 - \frac{b_t}{b_t - a_t}\Big) \sum_{\delta_k \leq a_t} w_k - \Big(\frac{1}{b_t - a_t}\Big) \sum_{a_t < \delta_k < b_t} w_k \delta_k - \Big(\frac{b_t}{b_t - a_t}\Big) \sum_{\delta_k \geq b_t} w_k.$$

Then, for $a = a_0$, $b = b_0$, as X is a $(p|p)$-centroid with $Y(X) = X$, we have that

$$\Delta_0 = W_B(X, Y) - W_B(X, X) = \sum_k w_k f_k(\delta_k) - \sum_k w_k f_k(0) =$$

$$\sum_k w_k f_k(\delta_k) - \frac{b_0}{b_0 - a_0} \sum_k w_k \leq 0$$

Consider now the difference $(b_1 - a_1)\Delta_1 - (b_0 - a_0)\Delta_0$. We obtain

$$(b_1 - a_1)\Delta_1 - (b_0 - a_0)\Delta_0 = \sum_{\delta_k \leq a_0} w_k(-a_1 + a_0) + \sum_{a_0 < \delta_k \leq a_1} w_k(-a_1 + \delta_k) +$$

$$\sum_{b_0 \leq \delta_k < b_1} w_k(-\delta_k + b_0) + \sum_{\delta_k \geq b_1} w_k(-b_1 + b_0).$$

Since $a_0 \leq a_1 \leq 0 < b_0 \leq b_1$, each addend in the previous summation is less than or equal to zero, and we have that $(b_1 - a_1)\Delta_1 - (b_0 - a_0)\Delta_0 \leq 0$, which implies,

$$\Delta_1 \leq \frac{b_0 - a_0}{b_1 - a_1} \Delta_0 \leq 0.$$

From $\Delta_1 \leq 0$ we conclude that, for a_1, b_1, $W_B(X, X) = \max_{Y \in L^p} W_B(X, Y)$. On the other hand, $W_B(X, X) = \dfrac{b_1}{b_1 - a_1} W_T$ is a lower bound of the optimum score. Therefore $S(X) = \min_{Z \in L^p} \max_{Y \in L^p} W_B(Z, Y)$.

□

Proposition 2. *Let, for every k,*

$$f_k(\delta) = f(\delta; a, b) = \begin{cases} 1 & \text{if} \quad \delta \leq a \\ 1 - \left(\frac{\delta - a}{b - a}\right)^q & \text{if} \quad a < \delta \leq b \\ 0 & \text{if} \quad \delta > b \end{cases}$$

where $a \leq 0 < b$ and q is an even number greater than or equal to 2. If, for $b = b_0$, set X with $Y(X) = X$ is a $(p|p)$-centroid, then for all $b \geq b_0$, we have $Y(X) = X$ and X is a $(p|p)$-centroid.

Proof. Similarly to Proposition 1.

Consider the customer choice rule defined by $f_k(\delta) = f(\delta; a, b_t)$, for all k and $t = 0, 1$, and suppose that, for $b = b_0$, X is a $(p|p)$-centroid and $Y(X) = X$. Let $b_1 \geq b_0$.

Using the same notation as in Proposition 1, for $t = 0, 1$ and for any $Y \in L^p$, we have

$$\Delta_t = W_B(X,Y) - W_B(X,X) =$$

$$\frac{a^q}{(b_t-a)^q}\sum_{\delta_k\leq a} w_k + \sum_{a<\delta_k<b_t} w_k\Big(- \frac{(\delta_k-a)^q}{(b_t-a)^q} + \frac{a^q}{(b_t-a)^q}\Big) + \Big(-1 + \frac{a^q}{(b_t-a)^q}\Big)\sum_{\delta_k\geq b_t} w_k.$$

Then

$$(b_1-a)^q\Delta_1 - (b_0-a)^q\Delta_0 =$$
$$\sum_{b_0\leq\delta_k<b_1} w_k\Big(-(\delta_k-a)^q + (b_0-a)^q\Big) + \sum_{\delta_k\geq b_1} w_k\Big(-(b_1-a)^q + (b_0-a)^q\Big).$$

Since $a \leq 0 < b_0 \leq b_1$, each addend in the previous summation is less than or equal to zero, and we obtain $(b_1-a)^q\Delta_1 - (b_0-a)^q\Delta_0 \leq 0$, which implies,

$$\Delta_1 \leq \Big(\frac{b_0-a}{b_1-a}\Big)^q \Delta_0 \leq 0.$$

As in Proposition 1, from $\Delta_1 \leq 0$ we conclude that, for $b = b_1$, $W_B(X,X) = S(X) = \min_{Z\in L^p}\max_{Y\in L^p} W_B(Z,Y)$.

□

Proposition 3. *Let, for every k,*

$$f_k(\delta) = f(\delta; a, b) = \begin{cases} 1 & \text{if} \quad \delta \leq a \\ \left(\frac{b-\delta}{b-a}\right)^q & \text{if} \quad a < \delta \leq b \\ 0 & \text{if} \quad \delta > b \end{cases}$$

where $a \leq 0 < b$ and q is an even number greater than or equal to 2. If, for $a = a_0$, set X with $Y(X) = X$ is a $(p|p)$-centroid, then for all a with $0 \geq a \geq a_0$, we have $Y(X) = X$ and X is a $(p|p)$-centroid .

Proof. Similarly to Propositions 1 and 2.

Consider the customer choice rule defined by $f_k(\delta) = f(\delta; a_t, b)$, for all k and $t = 0, 1$. For $a = a_0$, let X be a $(p|p)$-centroid with $Y(X) = X$. Suppose $0 \geq a_1 > a_0$. In this case, for $t = 0, 1$ and for any $Y \in L^p$, we have

$$\Delta_t = W_B(X,Y) - W_B(X,X) =$$

$$\sum_{\delta_k \le a_t} w_k\Big(1 - \frac{b^q}{(b-a_t)^q}\Big) + \sum_{a_t<\delta_k<b} w_k\Big(\frac{(b-\delta_k)^q}{(b-a_t)^q} - \frac{b^q}{(b-a_t)^q}\Big) + \sum_{\delta_k \ge b} w_k\Big(- \frac{b^q}{(b-a_t)^q}\Big).$$

We obtain

$$(b-a_1)^q\Delta_1 - (b-a_0)^q\Delta_0 =$$
$$\sum_{\delta_k \le a_0} w_k\Big((b-a_1)^q - (b-a_0)^q\Big) + \sum_{a_0<\delta_k\le a_1} w_k\Big((b-a_1)^q - (b-\delta_k)^q\Big).$$

From $0 \ge a_1 > a_0$ it follows that $(b-a_1)^q - (b-a_0)^q < 0$. From $a_0 < \delta_k \le a_1 < 0$ it follows that $(b-a_1)^q - (b-\delta_k)^q \le 0$. Therefore $(b-a_1)^q\Delta_1 - (b-a_0)^q\Delta_0 \le 0$, as $\Delta_0 \le 0$, it implies $\Delta_1 \le 0$, and we conclude that $S(X) = W_B(X,X) = \min_{Z\in L^p}\max_{Y\in L^p} W_B(Z,Y)$. □

Proposition 4. *Consider the piecewise linear functions*

$$f_k(\delta) = f(\delta; a_k, b_k) = \begin{cases} 1 & if \quad \delta \le a_k \\ \frac{b_k-\delta}{b_k-a_k} & if \quad a_k < \delta \le b_k \\ 0 & if \quad \delta > b_k \end{cases}$$

where $a_k \le 0 < b_k$, for all k. Let X_M^w be the weighted p-median where demand point k has a weight equal to $\dfrac{1}{b_k - a_k}$. If $a_k \le d_{ki} - d_{kX_M^w} \le b_k$ for all $k \in K$, $i \in I$, then the weighted p-median, X_M^w, is a $(p|p)$-centroid and a $(p|X_M^w)$-medianoid, and the optimum score is $S(X_M^w) = \sum_k \dfrac{b_k}{b_k - a_k} w_k$.

Proof. If $a_k \le d_{ki} - d_{kX_M^w} \le b_k$ for all $k \in K$, $i \in I$, then for all $Y \in L^p$, $a_k \le d_{kY} - d_{kX_M^w} \le b_k$ for all $k \in K$. Therefore $W_B(X_M^w, Y) = \sum_k w_k \dfrac{b_k - \delta_k}{b_k - a_k}$, where $\delta_k = d_{kY} - d_{kX_M^w}$, and we have

$$W_B(X_M^w, Y) = \sum_k w_k \frac{b_k - \delta_k}{b_k - a_k} = \sum_k w_k \frac{b_k}{b_k - a_k} - \sum_k w_k \frac{\delta_k}{b_k - a_k} =$$
$$\sum_k w_k \frac{b_k}{b_k - a_k} - \Big(\sum_k w_k \frac{d_{kY}}{b_k - a_k} - \sum_k w_k \frac{d_{kX_M^w}}{b_k - a_k}\Big) =$$
$$W(X_M^w, X_M^w) - \Big(\sum_k w_k \frac{d_{kY}}{b_k - a_k} - \sum_k w_k \frac{d_{kX_M^w}}{b_k - a_k}\Big).$$

From

$$\sum_k w_k \frac{d_{kX_M^w}}{b_k - a_k} = \min_{Z\in L^p} \sum_k w_k \frac{d_{kZ}}{b_k - a_k}$$

it follows that

$$\sum_k w_k \frac{d_{kY}}{b_k - a_k} - \sum_k w_k \frac{d_{kX_M^w}}{b_k - a_k} \geq 0$$

and it is zero if $Y = X_M^w$. Therefore, for all $Y \in L^p$, $W(X_M^w, Y) \leq W(X_M^w, X_M^w)$ and we conclude that X_M^w is an $(X_M^w|p)$-medianoid. Moreover, as $W(X_M^w, X_M^w)$ is a lower bound of the optimal score, we conclude that $W(X_M^w, X_M^w) = \min_{X\in L^p} \max_{Y\in L^p} W(X, Y)$. That is, X_M^w is a $(p|p)$-centroid.

□

Corollary 1. *Consider the piecewise linear functions $f_k(d) = f(d; a_k, b_k)$ defined in Proposition 4, where $a_k \leq 0 < b_k$. If $\max_{i,k} d_{ki} \leq \min\{-a_k, b_k\}$ for all k, then the weighted p-median, X_M^w, is a $(p|p)$-centroid and a $(p|X_M^w)$-medianoid, and the optimum score is $S(X_M^w) = \sum_k \frac{b_k}{b_k - a_k} w_k$.*

Proof. If $\max_{i,k} d_{ki} \leq \min\{-a_k, b_k\}$ then, for all $X, Y \in L^p$ and all k, we have $a_k \leq d_{kY} - d_{kX} \leq b_k$. The result follows from Proposition 4.

□

Corollary 2. *Consider the piecewise linear functions $f_k(d) = f(d; a_k, b_k)$ defined in Proposition 4, where $a_k = a$ and $b_k = b$ for all k, with $a \leq 0 < b$. If $\max_{i,k} d_{ki} \leq \min\{-a, b\}$, then the p-median, X_M, is a $(p|p)$-centroid and a $(p|X_M)$-medianoid, and the optimum score is $S(X_M) = \frac{b}{b-a} W_T$ where W_T is the total demand. In particular, for the symmetric case, $a = -b$, $S(X_M) = \frac{1}{2} W_T$.*

Proof. It follows from Proposition 4.

□

3. An example

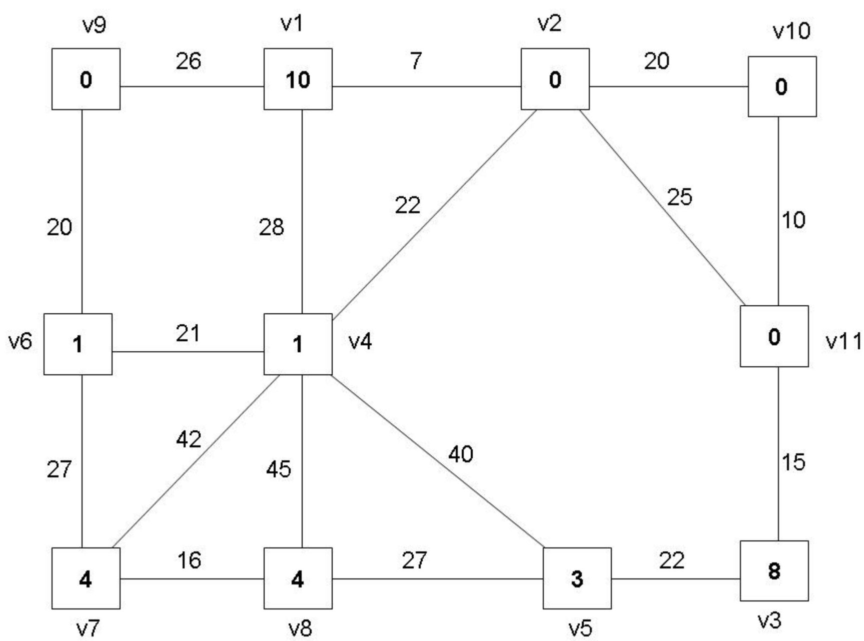
Consider the network represented in Figure 2, where the lengths are indicated beside the edges and the demand at each node is shown inside the squares. Table 2 shows the results for $r = p = 2$, using the binary choice rule, piecewise linear, concave and convex functions described in Table 1. Three pair of values for a and b are considered: (1) $-a = b = 2.86$, (2) $a = -2.86$, $b = 40$, and (3) $a = -40$, $b = 2.86$. For each pair of values, two cases are solved: (a) no restrictions, coincident locations are allowed, and (b) coincident locations are not allowed. The first column indicates the customer choice rule used to obtain the locations. The second and fourth columns show the optimal locations for each case. Column 6 shows the demand captured by the follower if the locations for leader and follower coincide. In some cases, multiple optimal solutions exist. The last row shows the results for a scenario in which different choice functions for different demand points are considered, and where symmetric piecewise linear, concave and convex decay functions are used.

The results obtained show that symmetric and asymmetric pro follower choice rules favour coincident locations, and so the follower opens one or both facilities at a point previously selected by the leader. For asymmetric pro leader decay functions, coincidences do not occur. For any X, $W_B(X, X)$ provides a lower bound of the score $S(X)$. The most advantageous scenario for the follower is given by the concave asymmetric pro follower decay function. From Corollary 2, we deduce that, in the linear case, for a sufficiently large $-a$ and b, the 2-median, $\{1, 5\}$, is a (2|2)-centroid and the optimum score is $31\frac{b}{b-a}$.

Table 2: W_B value and optimal locations (in some cases there are multiple solutions)

Rule	Coincidences allowed		Coincidences not allowed		$X = Y$
	$X; Y$	$W_B = S(X)$	$X; Y$	$W_B = S(X)$	W_B
Binary ($\mu = 0.5$)	1,7; 1,3	16.50	1,3; 5,6	13	15.50
$-a = b = 2.86$ (Symmetric)					
Linear	1,7; 1,11	16.50	1,3; 5,6	13	15.50
Concave	1,3; 1,3	23.25	1,3; 5,6	13	23.25
Convex	1,3; 1,5	14	1,3; 5,6	13	7.75
$a = -2.86$, $b = 40$ (Asymmetric pro follower)					
Linear	1,3; 1,3	28.93	1,5; 2,8	22.72	28.93
Concave	1,3; 1,3	30.86	1,2; 5,9	26.23	30.86
Convex	1,3; 1,3	27.00	1,5; 2,8	18.31	27.00
$a = -40$, $b = 2.86$ (Asymmetric pro leader)					
Linear	1,3; 5,8	9.88	1,3; 5,8	9.88	2.07
Concave	1,3; 5,6	11.16	1,3; 5,6	11.16	0.14
Convex	2,5; 3,7	7.30	2,5; 3,7	7.30	4.00
Symmetric mixed scenario					
Linear-Concave-Convex	1,5; 3,8	14.77	1,5; 3,8	14.77	14

Figure 2: Network of Example



4. Linear formulations

4.1. The $(r|X_p)$*-medianoid problem*

The problem of the follower is the $(r|X_p)$-medianoid problem. Given the position of the leader, the follower wishes to open facilities at the locations which provide the maximum market share. In order to formulate this problem, we introduce the following variables

$$y_i = \begin{cases} 1 & if \quad \text{the follower opens a facility at point } l_i \\ 0 & \quad\text{otherwise} \end{cases} \tag{4}$$

$$z_{ki} = \begin{cases} 1 & if \quad \text{client } c_k \text{ visits a facility at } l_i \\ 0 & \quad\text{otherwise} \end{cases} \tag{5}$$

for $i \in [1..m]$, $k \in [1..n]$.

The problem can be formulated as follows

$$\begin{aligned}
&\max \sum_{i=1}^{m} \sum_{k=1}^{n} h_{ki} z_{ki} \\
&\text{subject to:} \\
&\sum_{i=1}^{m} y_i = r \\
&\sum_{i=1}^{m} z_{ki} \leq 1 && k \in [1..n] \\
&z_{ki} \leq y_i && i \in [1..m],\ k \in [1..n] \\
&z_{ki},\ y_i \in \{0,1\} && i \in [1..m],\ k \in [1..n]
\end{aligned} \tag{6}$$

where $h_{ki} = w_k f_k(\delta_{ki})$ and $\delta_{ki} = d_{ki} - d_{kX_p}$, $i \in [1..m]$, $k \in [1..n]$.

The objective function represents the demand captured by the follower. The first constraint states that the follower opens r facilities. Constraint $\sum_{i=1}^{m} z_{ki} \leq 1$ means that each client k (demand point k) patronizes at most one follower's facility. Constraint $x_{ki} \leq y_i$ indicates that a client k patronizes a facility at location l_i only if a facility is open at that point. This model has $m(1+n)$ variables and $1+n(1+m)$ constraints. Observe that the integrality constraints for z_{ki} can be relaxed, in which case z_{ki} is interpreted as the proportion of demand at c_k captured by the follower at location l_i.

This optimization problem has good properties with respect to the performance of the greedy algorithm (which provides a lower bound) and the relaxed linear programming problem (which gives an upper bound). An alternative formulation with a lower dimension is described in Berman and Krass (2002). For any k, consider μ_{kj}, $j \in [1..m_k]$, the nonzero values of h_{ki} with μ_{kj} indexed in increasing order. For $k \in [1..n]$ and $j \in [1..m_k]$, define the sets

$$\begin{aligned} L_{kj} &= \{l_i \in L : h_{ki} = \mu_{kj}\} \\ I_{kj} &= \{i \in [1..m] : l_i \in L_{kj}\}. \end{aligned} \tag{7}$$

Set L_{kj} contains the locations in L that capture the client c_k at level μ_{kj}. Set I_{kj} is the set of indexes of points in L_{kj}.

Now $z_{kj} = 1$ if client c_k is captured by a follower's facility at level μ_{kj} and $z_{kj} = 0$ otherwise. Then, the $(r|X_p)$-medianoid problem can be formulated as follows

$$\begin{aligned} &\max \sum_{k=1}^{n} \sum_{j=1}^{m_k} \mu_{kj} z_{kj} \\ &\text{subject to:} \\ &\sum_{i=1}^{m} y_i = r \\ &\sum_{j=1}^{m_k} z_{kj} \leq 1 && k \in [1..n] \\ &z_{kj} \leq \sum_{i \in I_{kj}} y_i && k \in [1..n],\ j \in [1..m_k] \\ &y_i,\ z_{kj} \in \{0,1\} && j \in [1..m_k],\ k \in [1..n] \end{aligned} \tag{8}$$

This model has $m + \sum_{k=1}^{n} m_k$ variables and $1 + n + \sum_{k=1}^{n} m_k$ constraints.

4.2. The $(r|p)$-centroid problem

The problem of the leader is the $(r|p)$-centroid problem. To formulate this problem we introduce the following variables

$$x_i = \begin{cases} 1 & if \quad \text{the leader opens a facility at point } l_i \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$u_{ki} = \begin{cases} 1 & if \quad \text{client } c_k \text{ visits a leader facility located at point } l_i \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $i \in [1..m]$, $k \in [1..n]$. Let $J = [1..\binom{m}{r}]$ be the index set corresponding to L^r. We can formulate the $(r|p)$-centroid problem as follows

$$\begin{array}{ll}
\min\ W & \\
\text{subject to:} & \\
\sum_{i=1}^{m} x_i = p & \\
\sum_{i=1}^{m} \sum_{k=1}^{n} h_{ki}^{j} u_{ki} \leq W & j \in J \\
\sum_{i=1}^{m} u_{ki} = 1 & k \in [1..n] \\
u_{ki} \leq x_i & i \in [1..m],\ k \in [1..n] \\
u_{ki},\ x_i \in \{0, 1\} & i \in [1..m],\ k \in [1..n]
\end{array} \tag{11}$$

where

$$h_{ki}^{j} = w_k f_k(\delta_{ki}^{j}) \text{ and } \delta_{ki}^{j} = d_k(Y_j) - d_{ki}. \tag{12}$$

Expression (12) represents the demand at c_k captured by the follower if he/she has facilities at Y_j and the closest leader's facility to client c_k is located at location l_i. Observe that constraint $\sum_{i=1}^{m} u_{ki} = 1$ implies that every client is assigned to a facility but the leader may not capture demand from this demand point. This model has $m(1 + n)$ variables and $1 + n(1 + m) + \binom{m}{r}$ constraints. Integrality constraints for u_{ki} can be relaxed.

5. An exact solution approach

Some exact procedures proposed in the literature to solve the binary discrete $(r|p)$-centroid (Alekseeva et al., 2010; Campos Rodríguez et al., 2010; Rodoredo and Pessoa, 2013) can be adapted to solve the discrete $(r|p)$-centroid for other customer choice functions. The following algorithm allows us to obtain optimal solutions for the $(r|p)$-centroid problem. The basic idea is to calculate lower and upper bounds of the optimum W^* until these two bounds coincide. At each iteration, Problem (11) is modified by replacing the set of follower's feasible solutions, L^r, by a subset $\mathcal{F} \subset L^r$. This problem is solved to obtain a solution X and the corresponding score $S_{\mathcal{F}}(X)$.

Algorithm:

Step 1 *Initialization.*

1.1 Select s feasible leader's solutions X_i, $i = 1, ..., s$. Solve the follower's problem for X_i, $i = 1, ..., s$. An upper bound of W^* is $\overline{W} = \min_i S(X_i)$. Let $X^* = X$ with $S(X) = \overline{W}$.

1.2 Let $\mathcal{F} = \{Y_i\}_{i=1}^q$ be the initial family of good follower candidates. Set $\underline{W} = 0$

Step 2 *Iterations.* Repeat, until $\underline{W} = \overline{W}$.

2.1 Solve the leader's problem using $\mathcal{F}$ instead L^r in (11). Let X be the optimal solution obtained.

If the optimal value obtained $S_{\mathcal{F}}(X)$ verifies $S_{\mathcal{F}}(X) > \underline{W}$ then do $\underline{W} = S_{\mathcal{F}}(X)$. If $\underline{W} = \overline{W}$, then $W^* = \underline{W} = \overline{W}$ is the optimal value and $X^* = X$ is the optimal set of locations for the leader.

2.2 Solve the follower's problem for X. If $S(X) < \overline{W}$ then set $\overline{W} = S(X)$ and $X^* = X$. If $\underline{W} = \overline{W}$, then $W^* = \underline{W} = \overline{W}$ is the optimal value and X^* is the optimal set of locations for the leader.

Set $\mathcal{F} = \mathcal{F} \cup \{Y(X)\}$, , where $Y(X)$ is the $(r|X)$-medianoid.

6. Computational example

We now apply the algorithm described in Section 5 to solve the $(r|p)$-centroid problem, or leader's problem, for continuous piecewise linear decay functions in symmetric and asymmetric cases. We also analyze the binary case, considering that ties can be solved by assigning half of the demand to each player (sharing coefficient to follower equal to $\mu = 0.5$). For the linear case, we choose the extremes of the interval, a and b, taking $a = -\alpha\times\rho$ and $b = \beta\times\rho$, where $\alpha,\ \beta > 0$ and ρ is the average of the absolute values $|d_{ki} - d_{kj}|$ with $k \in K$, $i, j \in I$ and $i \neq j$. In the computational examples, we take $\alpha, \beta = 0.10,\ 0.25,\ 0.50$.

We take the data used in Alekseeva et al. (2010), where $C = L$, $n = m = 100$, and the points are chosen at random with a uniform distribution in a square measuring 7000×7000. Two cases for the demand are considered, case (a) $w_k = 1$ for all k (instances 1 to 10), and (b) the demand is generated by a uniform distribution in $[0, 200]$ (instances 11 to 20).

In order to limit the computational cost, we introduce a stop rule to stipulate the maximum number of iterations. The algorithm stops when (a) $\overline{W} - \underline{W} \leq \gamma$, where γ is a small number, and where $\underline{W}$ and $\overline{W}$ are, respectively, a lower and upper bounds of the optimum W^*; or (b) the maximum number of iterations is reached. If $\underline{W} = \overline{W}$ the optimum is obtained. If $p = r$ a lower bound is obtained when we consider the same locations for both competitors. In the computational examples γ is 0.1% of the total demand.

As the two initial leader's solutions, we consider the p-median and the solution obtained via the alternating algorithm. In this approach, each player (leader and follower) reacts to the actions of the competitor by choosing the locations which maximize its profit, that is, both players behave as followers. An alternating heuristic is used in Bhadury et al. (2003) to solve the centroid problem in the plane. This is also used in Alekseeva et al. (2010) to find the initial leader's solutions in a discrete space. The initial family of follower's solutions is composed of the $(r|X)$- medianoid when X is the p-median, the r-median, and the $(r|X)$-medianoid when X is the solution obtained via the alternating heuristic.

The computational results are shown in Tables 3 to 7, each of which corresponds to specific values of α and β. The first column represents the instance, the AH column shows the W value provided by the alternating heuristic and the W^* column shows the best W value obtained with the exact algorithm modified with the introduction of the above-described stop rule. The percentages of demand captured by the follower are shown in brackets. Column %Error shows, for each instance, the values $100 \times \dfrac{\overline{W} - \underline{W}}{W_T}$, where W_T represents the total demand. Column Iter1 shows the iteration in which the best objective value was found. Column Iter2 shows the iteration at which the algorithm stopped. The last column shows the elapsed time in seconds consumed to find the best solution.

These computational results were obtained using a PC Intel(R)Core(TM) i7-2700K CPU 3.50GHz, RAM 16GB. The solutions were obtained using CPLEX solver in GAMS.

The results for $p = r = 5$ and for the different scenarios are shown in Subsections 6.1 (for continuous piecewise functions) and 6.2 (for the binary choice rule). Subsection 6.3 shows, for several values of $p = r$, the percentage of demand captured by the follower and the iteration at which the best W value was reached in the piecewise linear case with $\alpha = 0.10$.

6.1. Results for piecewise linear decay functions

Tables 3, 4 and 5, show the results for the symmetric case with $\alpha = \beta = 0.10,\ 0.25,\ 0.50$, respectively. We observe that, in the symmetric case, W^* decreases when α increases (except for instance 11 and $\alpha = 0.25$ where %Error= 0.27). Although it is not presented in a table, the case $\alpha = \beta = 0.75$ was also solved. In this scenario, the optimum score for all instances corresponds to coincident locations, that is, leader and follower capture 50% of total demand. In the tables, the highest value of %Error is indicated in bold. In no case is this value higher than 0.33 (in other words, $\overline{W} - \underline{W}$ is at most 0.33% of total demand). The number of iterations required to find the best solution (Iter1) decreases when $\alpha = \beta$ increases, that is, when the *sharing interval increases*. According to Corollary 2, for sufficiently large values of α, the 5-median is a (5|5)-centroid. For low values of $\alpha = \beta$, some instances requires great computational effort, due to the difficulty of solving the problem of the leader at each iteration. For $\alpha = \beta = 0.10$, the AH solution was improved for all instances except for instance 20. For $\alpha = \beta = 0.25$, an improvement was obtained for 13 of the 20 instances, while for $\alpha = \beta = 0.50$ the AH solution was the best solution found in 17 of the 20 instances.

In the asymmetric case, with $\alpha = 0.10$, $\beta = 0.25$, for all instances, the solution is the (5|5)-median, which was obtained at iteration Iter1= 0. In this case, leader and follower open facilities at the same locations. Therefore, according to Proposition 1, for $\alpha \leq 0.10$ and $\beta \geq 0.25$, this solution is optimal. Table 6 shows the results for $\alpha = 0.25$, $\beta = 0.10$, in this case the maximum number of iterations allowed is 50. The AH solution was improved in all cases except for instance 4.

Table 3: Piecewise linear $\alpha = \beta = 0.10$

Instance	Linear $\alpha = \beta = 0.10$					
$p = r = 5$	AH	W^*	%Error	Iter1	Iter2	Time (sc)
1	52.503	51.808	0.03	22	22	203.297
2	53.691	52.241	0.11	23	100	326.995
3	53.849	53.136	0.10	33	100	1999.205
4	52.871	52.442	0.12	22	100	534.764
5	53.453	52.737	0.10	32	100	970.432
6	51.952	51.154	0	13	13	108.840
7	55.220	52.105	0.12	25	100	465.287
8	52.775	52.109	0.18	17	100	178.092
9	52.842	52.080	0.17	18	100	263.202
10	53.113	52.816	0.12	11	100	58.405
11	4407.669	4397.829 (50.61%)	0.18	8	100	23.448
12	5720.181	5618.586 (53.41%)	0.08	42	70	4576.865
13	5068.399	4897.234 (52.37%)	0.03	9	33	152.631
14	5175.613	5144.412 (51.82%)	0.11	13	100	102.367
15	5632.190	5485.154 (53.55%)	0.05	35	49	1329.587
16	5206.571	5030.136 (52.85%)	0.06	37	41	1462.280
17	6065.238	5920.656 (52.87%)	0.09	38	40	1381.901
18	5171.269	5066.573 (53.01%)	0.08	10	41	53.009
19	5707.178	5532.966(53.22%)	0.07	45	57	5980.559
20	5232.133	5232.133(51.16%)	**0.26**	0	100	0

In order to illustrate the results, we present some figures corresponding to instances 3 and 13. In these two instances, the demand points are the same but the distribution of the demand is different. Figure 3 represents the distribution of the demand for instance 13. Each demand point k is represented by a circle whose radius is proportional to its demand. In Figures 4 to 10, the leader's locations are represented by squares and those of the follower, by asterisks. Figure 4 shows the solution for instance 3. In this case, the demand for each client is equal to one and $\alpha = \beta = 0.10$. Figures 5, 6 and 7 show the solution for instance 13 in the symmetric case with $\alpha = 0.10$, $\alpha = 0.25$, and $\alpha = 0.50$, respectively. Observe that for instances 3 and 13, and for $\alpha = 0.10$ a coincidence occurs but

Table 4: Piecewise linear $\alpha = \beta = 0.25$

Instance	Linear $\alpha = \beta = 0.25$					
$p = r = 5$	AH	W^*	%Error	Iter1	Iter2	Time (sc)
1	51.721	51.473	0.11	4	100	10.241
2	51.183	50.680	0.03	4	4	9.657
3	52.085	51.437	0.17	16	100	189.684
4	50.633	50.227	0	6	6	18.789
5	51.305	51.305	0.14	0	100	0
6	51.035	50.339	0.06	3	3	7.483
7	51.582	51.490	0.10	11	100	71.293
8	51.427	50.880	0	6	6	47.95
9	50.965	50.965	0.17	0	100	0
10	51.212	51.212	0.13	0	100	0
11	4415.994	4415.994(50.82%)	0.27	0	100	0
12	5413.952	5413.952 (51.46%)	0.10	0	19	0
13	4732.596	4694.718 (50.20%)	0.05	2	3	8.163
14	5082.924	5058.789 (50.96%)	0.10	9	10	40.139
15	5249.534	5249.534 (51.25%)	0.14	0	100	0
16	4947.429	4929.889 (51.79%)	0	24	24	509.169
17	5834.244	5755.125 (51.39%)	0	17	17	153.789
18	4909.392	4899.036 (51.26%)	0.09	11	12	57.660
19	5419.721	5412.545 (52.06%)	0.08	17	22	363.160
20	5147.021	5147.021 (50.33%)	**0.33**	0	100	0

Table 5: Piecewise linear $\alpha = \beta = 0.50$

Instance	Linear $\alpha = \beta = 0.50$					
$p = r = 5$	AH	W^*	%Error	Iter1	Iter2	Time (sc)
1	50.334	50.334	0.18	0	100	0
2	50.032	50.032	0.03	0	0	0
3	50.178	50.178	0.18	0	100	0
4	50.045	50.045	0.05	0	0	0
5	50.392	50.392	**0.32**	0	100	0
6	50.047	50.047	0.05	0	100	0
7	50.121	50.121	0.12	0	100	0
8	50.111	50.111	0.11	0	0	0
9	50.000	50.000	0	0	100	0
10	50.723	50.723	0.22	0	100	0
11	4362.673	4362.673 (50.21%)	0.21	0	0	0
12	5268.607	5268.607 (50.08%)	0.08	0	0	0
13	4675.500	4675.500 (50%)	0	0	0	0
14	4963.500	4963.500 (50%)	0	0	0	0
15	5185.124	5144.842 (50.23%)	0.15	0	100	0
16	4787.602	4774.220 (50.16%)	0	0	2	0
17	5599.500	5599.500 (50%)	0	0	0	0
18	4830.291	4830.291 (50.54%)	0.07	0	2	0
19	5365.244	5353.006 (51.49%)	0.07	1	8	1.881
20	5116.786	5116.786 (50.04%)	0.04	0	0	0

Table 6: Piecewise linear $\alpha = 0.25,\ \beta = 0.10$

Instance	Linear $\alpha = 0.25$ $\beta = 0.10$					
$p = r = 5$	AH	W^*	%Error	Iter1	Iter2	Time (sc)
1	44.688	43.946	0.19	40	50	2280.811
2	43.301	43.301	0.12	0	50	0
3	45.979	44.739	0.14	20	50	858.145
4	45.684	43.473	0.12	30	50	1023.665
5	45.549	45.074	0.23	44	50	3561.767
6	43.965	42.531	0.08	38	38	1871.418
7	47.555	44.808	0.26	46	50	3694.808
8	46.873	44.297	0.11	43	50	2186.552
9	46.080	44.830	0.11	42	50	2947.284
10	45.189	44.070	0.15	45	50	2470.528
11	3829.513	3817.158 (43.93%)	0.10	10	38	52.877
12	4980.316	4860.531 (46.20%)	0.85	24	50	1007.607
13	4187.448	4110.376 (43.96%)	0.09	20	31	477.543
14	4291.753	4261.193 (42.92%)	0	25	25	844.550
15	4713.435	4544.743 (44.37%)	0.18	17	50	368.885
16	4468.938	4333.618 (45.53%)	1.08	7	50	56.068
17	5090.326	5015.803 (44.79%)	0.27	21	50	490.766
18	4178.370	4126.638 (43.18%)	0.07	27	34	1109.047
19	4929.027	4867.981 (46.83%)	**1.58**	5	50	21.419
20	4586.127	4511.942 (44.12%)	0.15	31	50	697.893

in different places. On the other hand, in Figures 5, 6 and 7, we see that as $\alpha = \beta$ increases the follower locates its facilities closer to the leader. For $\alpha = \beta = 0.50$ both competitors choose the same locations, a 5-median. Figure 8 shows the solution obtained for $\alpha = 0.25$, $\beta = 0.10$ (asymmetric in favour of the leader). Observe that for $\alpha = 0.25, \beta = 0.10$, leader and follower choose different locations. For $\alpha = 0.10, \beta = 0.25$ (asymmetric in favour of the follower), the solution for leader and follower is a 5-median.

6.2. Results for the binary case

In this subsection we present some results for the binary rule for $p = r = 5$. Table 7 shows the results for the case in which coincident locations are allowed and where, in case of tie ($d_{kY} = d_{kX}$), the follower captures $\mu \times w_k$, where $0 \leq \mu \leq 1$. We consider $\mu = 0.5$. When $p = r$, a lower bound of the optimal capture for the follower is $0.5W_T$. The last column of the table shows the optimal value, W_0^*, for the binary case oriented to the leader ($\mu = 0$), these values having been taken from Alekseeva et al. (2010).

In this scenario, for all instances, the best W value was obtained before 50 iterations, the greatest computational effort was required for instance 12, when 44 iterations were required to obtain the best solution. If we compare this with the results for the symmetric linear case when $\alpha = \beta = 0.10$, we see that, except for instances 11 and 20, the upper bound of the error (column %Error) for the binary case is significantly higher, as a consequence of the discontinuity of the binary decay function.

Figure 3: Demand distribution. Instance 13

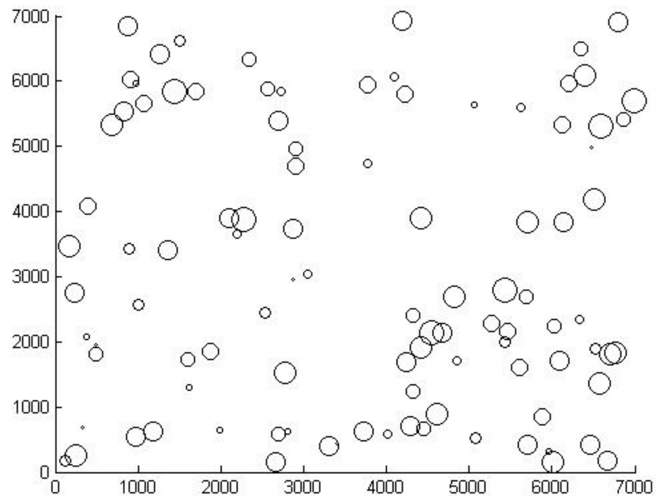


Figure 4: Linear $\alpha = \beta = 0.10$. Instance 3

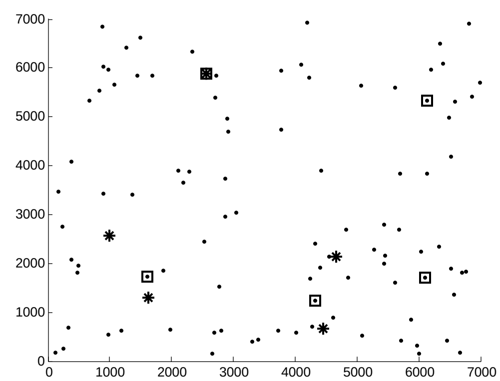


Figure 5: Linear $\alpha = \beta = 0.10$. Instance 13

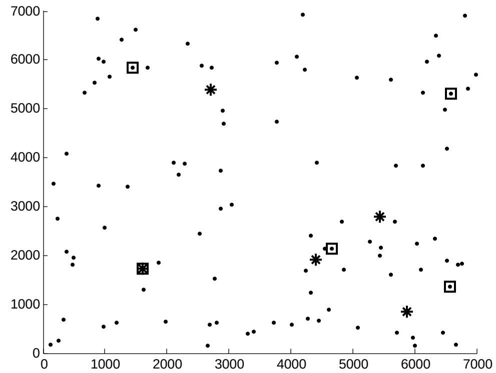


Figure 6: Linear $\alpha = \beta = 0.25$. Instance 13

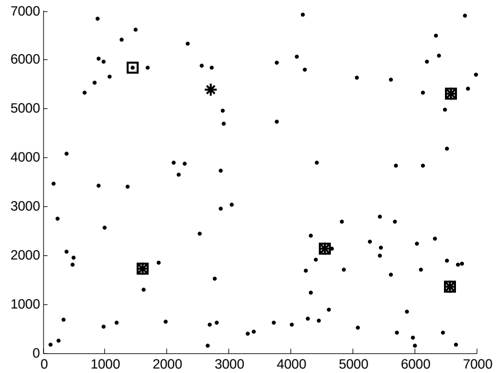


Figure 7: Linear $\alpha = \beta = 0.50$. Instance 13

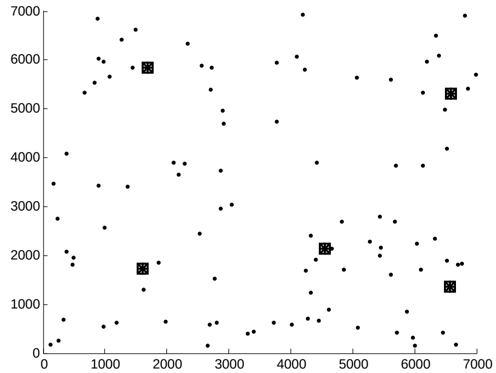


Figure 8: Linear $\alpha = 0.25\ \beta = 0.10$. Instance 13

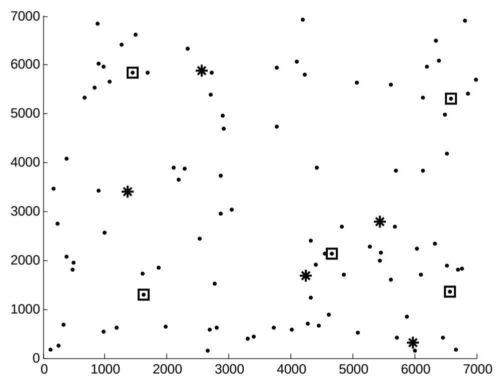


Table 7: Binary $\mu = 0.50$

Instance	Binary $\mu = 0.50$						
$p = r = 5$	AH	W^*	%Error	Iter1	Iter2	Time (sc)	W_0^*
1	54	53	0.5	9	50	49.884	53
2	54	53	0.5	10	50	50.449	52
3	56	55	1	4	50	21.050	55
4	55	54	1	34	50	1318.348	53
5	55	53	0.5	11	50	109.520	53
6	54	53	0.5	24	50	518.872	53
7	55	53	0.5	22	50	489.855	53
8	56	53	0.5	32	50	961.745	52
9	53	53	0.5	0	50	0	53
10	54	54	0.5	0	50	0	53
11	4847	4550 (52.36%)	0.098	10	31	43.390	4550 (52.36%)
12	5929	5698 (54.16%)	1.188	44	50	3002.127	5698 (54.16%)
13	5321	5222 (55.85%)	2.529	24	50	1325.695	5136 (54.92%)
14	5335	5335 (53.74%)	0.524	0	50	0	5249 (52.88%)
15	5776	5675 (55.40%)	1.611	29	50	1191.760	5649 (55.15%)
16	5274	5173.5 (54.35%)	0.888	24	50	652.883	5025 (52.79%)
17	6333	6046 (53.99%)	0.455	33	50	986.178	6046 (53.99%)
18	5232	5153 (53.92%)	0.544	30	50	912.467	5153(53.92%)
19	5975	5696 (54.79%)	1.462	29	50	2476.411	5696 (54.79%)
20	5655	5392.5 (52.73%)	0	34	34	727.762	5303 (51.86%)

Figure 9: Binary $\mu = 0$ (oriented to leader). Instance 13

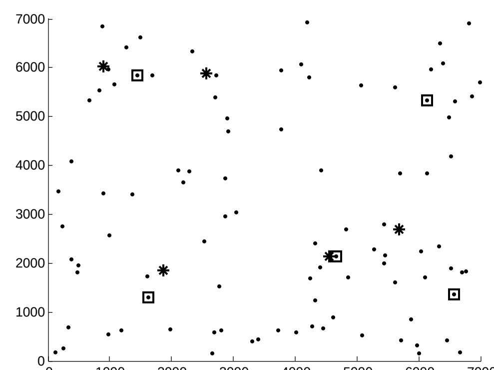
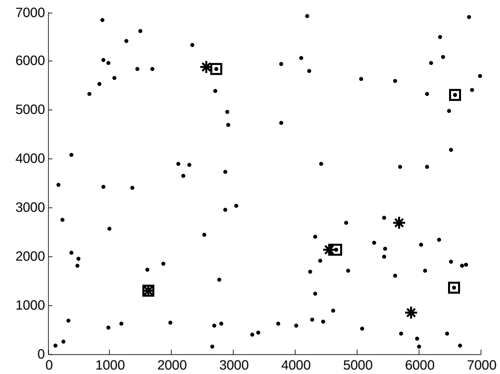


Figure 10: Binary $\mu = 0.5$. Instance 13

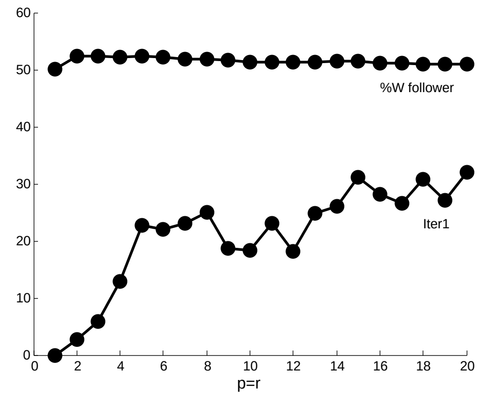


6.3. *Comparative analysis for different scenarios and values of p*

In the results presented in Sections 6.1 and 6.2 for $p = r = 5$, we observed that there are three locations which appear in all the leader solutions for the symmetric scenarios $\alpha = \beta = 0.10, 0.25, 0.50$, and the asymmetric case $\alpha = 0.10$, $\beta = 0.25$. Two of these points appear in all scenarios except that of the binary oriented to the leader. One of these points appears in the solution for all scenarios analyzed. Other points coincide in the solution of three of these scenarios. This finding suggests that there are points which are *good* locations for the leader in most cases.

Figure 11 shows the average demand captured by the follower (as a percentage) and the Iter1 value for $1 \leq p = r \leq 20$ in the symmetric linear case with $\alpha = 0.10$. As the follower can open facilities at the same locations as the leader, the percentage of demand captured is always greater than or equal to 50%. For the scenarios analyzed, this percentage is less than 52.4% in all cases. The number of iterations required to reach the best objective value is always less than 32. The error (percentage), defined as $100 \times \frac{\overline{W}-W}{W_T}$, is less than 0.6 in all cases, with lower values for small values of p. The average error is 0.343%. As p increases, so does the number of coincident locations, while the percentage of demand captured by the follower tends to 50%. This behaviour of the demand captured by the follower differs from that observed for the binary case oriented to the leader; in this binary scenario, the market share of the follower for the highest values of p is significantly less than 50% (Alekseeva et al. 2010).

Figure 11: Symmetric case $\alpha = \beta = 0.10$



7. Conclusions

In this paper we generalize the discrete $(r|p)$-centroid problem to consider customer choice rules defined by generic decay functions. A customer may visit the closest facility of each of two competitors, leader and follower, using at each of these facilities an amount of buying power which depends on the difference in travel distance (or time) to the closest competing facilities. For $p = r$, we obtain interesting theoretical results for piecewise linear, concave and convex decay functions. In particular, for continuous piecewise linear functions, we prove that under certain conditions, the p-median is a $(p|p)$-centroid and the optimal score corresponds to coincident locations, that is, leader and follower open their facilities at the same places.

For particular piecewise linear decay functions, our computational examples show that when the sharing zone is expanded the follower tends to locate facilities closer to those of the leader and the optimal score decreases. The solution consisting of coincident locations provides a lower bound of the optimum, while an upper bound was obtained via an alternating heuristic. A comparison of the results for $1 \leq p = r \leq 20$ suggests that when $p = r$ increases the demand captured by the follower tends to 50% of the total demand.

To obtain the solutions presented, we applied an exact procedure which requires the resolution of a constrained leader's problem at each iteration. As the number of iterations increases, this problem becomes more complex and the computational effort required increases significantly. This outcome suggests that heuristic procedures should be used to solve the $(r|p)$-centroid problem.

References

[1] Alekseeva, E., Kochetov, Y., 2013. Matheuristics and exact methods for the discrete $(r|p)$-centroid problem. In: Talbi, E. (Eds.), *Metaheuristics for Bi-level Optimization, Studies in Computational Intelligence 482*. Springer-Verlag, Berlin Heidelberg. pp. 189-220.

[2] Alekseeva, E., Kochetova, N., Kochetov, Y., Plyasunov, A., 2010. Heuristic and exact methods for the discrete $(r|p)$-centroid problem. In: Cowling, P.I., Merz, P. (Eds.), *EvoCOP'10 Proceedings of the 10th European Conference on Evolutionary Computation in Combinatorial Optimization*. Springer-Verlag, Berlin Heidelberg. pp. 11-22.

[3] Benati, S., Laporte, G., 1994. Tabu search algorithms for the $(r|X p)$-medianoid and $(r|p)$-centroid problems. *Location Science* 2, 193-204.

[4] Berman, O., Drezner, Z., Krass, D., 2010. Generalized coverage: New developments in covering location models. *Computers & Operations Research* 37, 1675-1687.

[5] Berman, O., Krass, D., 2002. The generalized maximal covering location problem. *Computers & Operations Research* 29, 563-581.

[6] Berman, O., Krass, D., Drezner, Z., 2003. The gradual covering decay location problem on a network. *European Journal of Operational Research* 151, 474-480.

[7] Bhadury, J., Eiselt, H.A., Jaramillo, J.A., 2003. An alternating heuristic for medianoid and centroid problems in the plane. *Computers* & *Operations Research* 30, 553-565.

[8] Biesinger, B., Hu, B., Raidl, G., 2015. Models and algorithms for competitive facility location problems with different customer behaviour. *Annals of Mathematics and Artificial Intellegence,* DOI 10.1007/s10472-014-9448-0.

[9] Biesinger, B., Hu, B., Raidl, G., 2015. A hybrid genetic algorithm with solution archive for the discrete $(r|p)$-centroid problem. *Journal of Heuristics* 21, 391-431.

[10] Campos Rodríguez, C.M., Santos Peñate, D.R., Moreno Pérez, J.A., 2010. An exact procedure and LP formulations for the leader-follower location problem. *TOP* 18, 97-121.

[11] Davydov, I.A., Kochetov, Y.A., Mladenovic, N., Urosevic, D., 2014. Fast metaheuristics for the discrete $(r|p)$-centroid problem. *Automation and Remote Control* 75, 677-687.

[12] Devletoglou, N.E., 1965. A dissenting view of duopoly and spatial competition. *Economica*, 141-160

[13] Devletoglou, N.E., Demetriou P.A., 1967. Choice and threshold: a further experiment in spatial duopoly. *Economica,* 351-371

[14] Gosh, A., Craig, C.S., 1984. A location-allocation model for facility planning in a competitive environment. *Geographical Analysis* 20, 39-51.

[15] Hakimi, S.L., 1990. Location with spatial interactions: competitive locations and games. In: Mirchandani, P.B., Francis, R.L. (Eds.), *Discrete Location Theory*. Wiley, New York. pp.439-478.

[16] Costa Roboredo, M., Alves Pessoa, A., 2013. A branch-and-bound algorithm for the discrete $(r|p)$-centroid problem. *European Journal of Operational Research* 224, 101-109.